

Quantum Public-Key Encryption with Information Theoretic Security

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Abstract

We propose a definition for the information theoretic security of a quantum public-key encryption scheme, and present bit-oriented and two-bit-oriented encryption schemes satisfying our security definition via the introduction of a new public-key algorithm structure. We extend the scheme to a multi-bit-oriented one, and conjecture that it is also information theoretically secure, depending directly on the structure of our new algorithm.

Keywords:

ciphertext indistinguishability, quantum public-key, information theoretic security

1. Introduction

The public-key encryption schemes currently used will not keep their security in the post-quantum era, so it is necessary to find new kinds of encryption to resist the attacks of quantum adversaries. Quantum public-key encryption (QPKE) is one solution, which has been studied for about ten years. Okamoto et al [1] put forward a knapsack-based scheme which involves a quantum algorithm during key generation. Gottesman and Chuang [2] were the first to use quantum states as public keys. Gottesman was also the first to put forward "Quantum Public Key Cryptography with Information-Theoretic Security" [3] based on Einstein-Podolsky-Rosen pairs. Yang [4] has discussed public-key encryption of quantum messages based on a classi-

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cal computational complexity hypothesis. Kawachi et al [5] investigated the cryptographic property "computational indistinguishability" of two quantum states generated via fully flipped permutations ($QSCD_{ff}$), and gave a QPKE scheme based on this. Nikolopoulos [6] suggested another scheme based on qubit rotations. The latter two schemes are bit-oriented, and Kawachi et al extended their scheme to multibits [7], which was later shown in [8] to have bounded information theoretic security.

2. Security of Quantum Public-Key Encryption

In classical public-key encryption (PKE), the ciphertext indistinguishability under a chosen plaintext attack (CPA) is defined as [9]: for every polynomial-size circuit family $\{C_n\}$, every positive polynomial $p(\cdot)$, all sufficiently large n , and every x, y in plaintext space, the probability $\Pr(\cdot)$ satisfies:

$$|\Pr[C_n(G_1(1^n), E_{G_1(1^n)}(x)) = 1] - \Pr[C_n(G_1(1^n), E_{G_1(1^n)}(y)) = 1]| < \frac{1}{p(n)}. \quad (1)$$

As the ciphertext is a quantum state in the quantum case, the ciphertext indistinguishability of QPKE is defined as the indistinguishability of any two quantum states in ciphertext space. Koshiba [10] extended indistinguishability to QPKE, but he restricted the circuit to a polynomial-size one. We propose here a definition of ciphertext indistinguishability of quantum public-key encryption beyond the computational complexity hypothesis.

Definition 1. *A quantum public-key encryption has ciphertext indistinguishability under CPA, if for every quantum circuit family $\{C_n\}$, for every positive polynomial $p(\cdot)$, all sufficiently large n , and every x, y in plaintext space, the probability $\Pr(\cdot)$ satisfies:*

$$|\Pr[C_n(G_1(1^n), E_{G_1(1^n)}(x)) = 1] - \Pr[C_n(G_1(1^n), E_{G_1(1^n)}(y)) = 1]| < \frac{1}{p(n)}. \quad (2)$$

where the encryption algorithm E is a quantum algorithm, and the ciphertexts $E(x)$ and $E(y)$ are quantum states.

The difference between our definition and Koshiba's is that there is no restriction on $\{C_n\}$ in our definition.

According to [9](see page 476), the definition we have presented here is related to information theoretic security under CPA. We define: A quantum public-key encryption is information theoretically secure under CPA if it satisfies Eq. (2).

In the following part, we give a quantum public-key encryption scheme which satisfies our definition of information theoretic security under CPA.

3. A bit oriented Public-key Encryption Scheme

Let: $\Omega_n = \{k \in Z_{2^n} \mid W_H(k) \text{ is odd}\}$ and $\Pi_n = \{k \in Z_{2^n} \mid W_H(k) \text{ is even}\}$, where $W_H(k)$ is k 's Hamming weight.

Definition 2. Define two n -qubit states:

$$\rho_{k,i}^0 = \frac{1}{2}(|i\rangle + |i \oplus k\rangle)(\langle i| + \langle i \oplus k|), \quad (3)$$

and

$$\rho_{k,i}^1 = \frac{1}{2}(|i\rangle - |i \oplus k\rangle)(\langle i| - \langle i \oplus k|), \quad (4)$$

where $i \in Z_{2^n}, k \in \Omega_n$.

The two states $\rho_{k,i}^0$ and $\rho_{k,i}^1$ can be generated effectively as follows:

For given i and k , use a permutation operator P_k on $|k\rangle$, so that $P_k|k\rangle = |1 \cdots 10 \cdots 0\rangle$. Let $P_k|i\rangle = |i'\rangle|i''\rangle$, so:

$$\begin{aligned} \frac{1}{\sqrt{2}}P_k(|i\rangle + |i \oplus k\rangle) &= \frac{1}{\sqrt{2}}(|i'\rangle|i''\rangle + |i' \oplus (2^{W_H(k)} - 1)\rangle|i'' \oplus 0\rangle) \\ &= \frac{1}{\sqrt{2}}(|i'\rangle + |\bar{i}'\rangle)|i''\rangle, \end{aligned} \quad (5)$$

where $|\bar{i}'\rangle$ is the state after applying the X operation on each qubit of $|i'\rangle$.

It can be seen that, $\frac{1}{\sqrt{2}}(|i'\rangle + |\bar{i}'\rangle)$ is a $W_H(k)$ bits GHZ state. Inverting the above process, we obtain an effective way to generate $\rho_{k,i}^0$ if GHZ states are available.

To produce $\rho_{k,i}^1$, we just apply Z on each qubit of $\rho_{k,i}^0$ for $W_H(k)$ odd. Then we have a polynomial-time quantum algorithm to convert $\rho_{k,i}^0$ to $\rho_{k,i}^1$ without k and i .

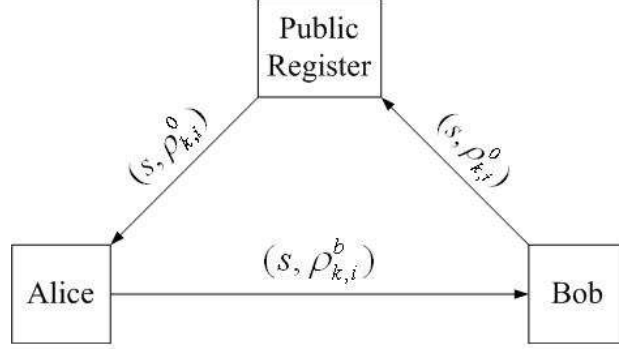


Figure 1: First, Bob sends his public-key $(s, \rho_{k,i}^0)$ to a public register. Alice gets Bob's public-key from the public register, then she encrypts b into $\rho_{k,i}^b$, and sends $(s, \rho_{k,i}^b)$ back to Bob.

3.1. Application in Quantum Public-Key Encryption

Our quantum public-key encryption is shown in Figure 1:

[Key Generation]

- (G1) Bob selects randomly a Boolean function $F : \Omega_n \rightarrow \Omega_n$ as private key;
- (G2) Bob selects $s \in \Omega_n$ randomly;
- (G3) Bob generates $\rho_{k,i}^0$, where $k = F(s)$, i is chosen randomly from Z_{2^n} ;
- (G4) Bob sends the classical-quantum pair $(s, \rho_{k,i}^0)$ to a public register as his public-key.

When Alice needs to send a classical bit b to Bob via the quantum channel, they can do as follows:

[Encryption]

- (E1) Alice gets one of Bob's public keys from the public register;
- (E2) Alice encrypts b into $\rho_{k,i}^b$, then sends $(s, \rho_{k,i}^b)$ to Bob;

[Decryption]

- (D1) Bob receives $(s, \rho_{k,i}^b)$, and calculates $k = F(s)$;
- (D2) Bob decrypts the one-bit message with k : $(s, \rho_{k,i}^b) \rightarrow b$.

Notes:

- (1) The Boolean function F can be chosen from a larger set $\{0, 1\}^{poly(n)}$, but when s is chosen, it should satisfy $F(s) \in \Omega_n$.
- (2) The public register should ensure that Alice obtains the correct public-key from Bob. This is a precondition of all public-key encryption schemes.

3.2. Trapdoor Property

Bob can decrypt the ciphertext states with k , but without i . So, we consider the mixed states: $\rho_k^0 = \frac{1}{2^n} \sum_i \rho_{k,i}^0$ and $\rho_k^1 = \frac{1}{2^n} \sum_i \rho_{k,i}^1$.

Lemma 3. *For given $k \in \Omega_n$, there exists a polynomial-time quantum algorithm that distinguishes ρ_k^0 and ρ_k^1 determinedly.*

Proof: Let ρ be the unknown state ρ_k^0 or ρ_k^1 , then the algorithm is given as follows:

- (1) Prepare two quantum registers, the first register holds a control bit in $|0\rangle\langle 0|$, and the second one holds ρ . After Hadamard's transformation is applied to the first register, the state of the system becomes:

$$\frac{(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)}{2} \otimes \rho. \quad (6)$$

- (2) Define Controlled- k operator C_k as: $C_k|0\rangle|i\rangle = |0\rangle|i\rangle$, $C_k|1\rangle|i\rangle = |1\rangle|i \oplus k\rangle$ for any $i \in Z_{2^n}$, (C_k can be realized via a group of CNOT operations), apply C_k to the two registers, then the result will be

$$\frac{1}{2^n} \sum_{i=0}^{2^n-1} |\varphi_{k,i}^0\rangle\langle\varphi_{k,i}^0|, \text{ if } \rho = \rho_k^0, \quad (7)$$

or

$$\frac{1}{2^n} \sum_{i=0}^{2^n-1} |\varphi_{k,i}^1\rangle\langle\varphi_{k,i}^1|, \text{ if } \rho = \rho_k^1, \quad (8)$$

where

$$\begin{aligned} |\varphi_{k,i}^0\rangle &= C_k[\frac{1}{2}(|0\rangle + |1\rangle)(|i\rangle + |i \oplus k\rangle)] \\ &= \frac{1}{2} [|0\rangle(|i\rangle + |i \oplus k\rangle) + |1\rangle(|i \oplus k\rangle + |i\rangle)], \end{aligned} \quad (9)$$

$$\begin{aligned}
|\varphi_{k,i}^1\rangle &= C_k \left[\frac{1}{2}(|0\rangle + |1\rangle)(|i\rangle - |i \oplus k\rangle) \right] \\
&= \frac{1}{2} [|0\rangle(|i\rangle - |i \oplus k\rangle) + |1\rangle(|i \oplus k\rangle - |i\rangle)]. \quad (10)
\end{aligned}$$

(3) Apply Hadamard transformation to the first register again:

$$(H \otimes I)|\varphi_{k,i}^0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|i\rangle + |i \oplus k\rangle), \quad (11)$$

$$(H \otimes I)|\varphi_{k,i}^1\rangle = \frac{1}{\sqrt{2}}|1\rangle(|i\rangle - |i \oplus k\rangle). \quad (12)$$

If $\rho = \rho_k^0$, the final state is $|0\rangle\langle 0| \otimes \rho_k^0$; if $\rho = \rho_k^1$, the final state is $|1\rangle\langle 1| \otimes \rho_k^1$.

It can be seen that we can distinguish ρ_k^0 and ρ_k^1 with correct probability 1 after measuring the first register. \square

3.3. Security Proof

If there exists an eavesdropper Eve between Alice and Bob, she may use two ways to attack the QPKE. One is to find information about k ; another is to distinguish between $\rho_{k,i}^0$ and $\rho_{k,i}^1$ to eavesdrop the message.

By measuring $\rho_{k,i}^0$ or $\rho_{k,i}^1$, Eve can get i or $i \oplus k$ with the same probability $1/2$, but she cannot get both of them. For each $\rho_{k,i}^0$ or $\rho_{k,i}^1$, i and s are chosen randomly, then $k = F(s)$ is also random. Although Eve may get $i \oplus k$ with probability $1/2$, she cannot obtain any information about k because she cannot get i at the same time. The security is the same as that of a one-time-pad in classical cryptography.

If Eve has an effective algorithm to distinguish $\rho_{k,i}^0$ and $\rho_{k,i}^1$ with non-negligible probability without k and i , that means Eve can distinguish the mixed states: $\rho_{odd}^0 = \frac{1}{2^{n-1} \cdot 2^n} \sum_{k \in \Omega_n} \sum_i \rho_{k,i}^0$, and $\rho_{odd}^1 = \frac{1}{2^{n-1} \cdot 2^n} \sum_{k \in \Omega_n} \sum_i \rho_{k,i}^1$. However, we have the following lemma:

Lemma 4. *The trace distance (defined as [11]) between ρ_{odd}^0 and ρ_{odd}^1 is $\frac{1}{2^{n-1}}$.*

Proof:

$$\rho_{odd}^0 - \rho_{odd}^1 = \frac{4}{2^{2n}} \sum_{k \in \Omega_n} \sum_i |i\rangle \langle i \oplus k| = \frac{4}{2^{2n}} A_{odd}, \quad (13)$$

where A_{odd} is a matrix with

$$a_{ij} = \begin{cases} 1, & W_H(i) \bmod 2 \neq W_H(j) \bmod 2 \\ 0, & W_H(i) \bmod 2 = W_H(j) \bmod 2 \end{cases}. \quad (14)$$

Applying an appropriate unitary operator to both sides of A_{odd} , we obtain

$$A' = \begin{bmatrix} 0 & 1 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 1 & 0 \\ & & \cdots & & \\ 0 & 1 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{\otimes(n-1)} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (15)$$

For $|A \otimes B| = |A| \otimes |B|$ and $tr(A \otimes B) = tr(A) \times tr(B)$, we have:

$$tr|A_{odd}| = tr|A'| = tr \left| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right|^{(n-1)} \times tr \left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right| = 2^n, \quad (16)$$

$$D(\rho_{odd}^0, \rho_{odd}^1) = \frac{1}{2} tr \left| \frac{4}{2^{2n}} A_{odd} \right| = \frac{4}{2 \cdot 2^{2n}} \cdot 2^n = \frac{1}{2^{n-1}}. \quad (17)$$

□

Theorem 5. *The quantum public-key encryption given above satisfies the inequality (2), so it has information theoretic security under CPA.*

Proof: For every quantum circuit family $\{C_n\}$, and every $x, y \in \{0, 1\}$,

$$\begin{aligned}
\Pr[C_n(G_1(1^n), E_{G_1(1^n)}(x)) = 1] &= \Pr[C_n(\rho_{k,i}^x) = 1] \\
&= \sum_{k,i} p_{k,i} \cdot \Pr[C_n(\rho_{k,i}^x) = 1] \\
&= \frac{1}{2^{2n-1}} \sum_{k,i} \Pr[C_n(\rho_{k,i}^x) = 1] \\
&= \Pr[C_n(\frac{1}{2^{2n-1}} \sum_{k,i} \rho_{k,i}^x) = 1] \\
&= \Pr[C_n(\rho_{odd}^x) = 1]. \tag{18}
\end{aligned}$$

If x and y are different, we consider the difference between ρ_{odd}^0 and ρ_{odd}^1 .

Any quantum circuit family $\{C_n\}$ that distinguishes between quantum states ρ_{odd}^0 and ρ_{odd}^1 can be regarded as distinguishing two probability distributions $\{p_m\}$ and $\{q_m\}$ based on a positive operator-valued measure (POVM) $\{E_m\}$ [12, 13], where $p_m = \text{tr}(C_n(\rho_{odd}^0)E_m)$ and $q_m = \text{tr}(C_n(\rho_{odd}^1)E_m)$ are the probability distributions of quantum measurement outcomes labeled by m . The maximum trace distance between $\{p_m\}$ and $\{q_m\}$ [11] over the whole set of POVMs determines the probability upper bound for distinguishing ρ_{odd}^0 and ρ_{odd}^1 by $\{C_n\}$,

$$\begin{aligned}
&|\Pr[C_n(\rho_{odd}^0) = 1] - \Pr[C_n(\rho_{odd}^1) = 1]| \\
&\leq \max_{\{E_m\}} \frac{1}{2} \sum_m |\text{tr}[E_m(C_n(\rho_{odd}^0) - C_n(\rho_{odd}^1))]| \\
&= \max_{\{E_m\}} D(p_m, q_m). \tag{19}
\end{aligned}$$

According to [11, 14],

$$\max_{\{E_m\}} D(p_m, q_m) = D(C_n(\rho_{odd}^0), C_n(\rho_{odd}^1)) \leq D(\rho_{odd}^0, \rho_{odd}^1) = \frac{1}{2^{n-1}}. \tag{20}$$

For any positive polynomial $p(\cdot)$, there exists a sufficiently large n so that Eq. (2) is satisfied. \square

Remark. Like the public-key encryption mentioned above, in [5, 7], the private-key π and public-key ρ_π^+ have a one-to-one correspondence, so the

public-key ρ_π^+ can only be used t times, $t = o(n \log n)$ [8]. Because π contains $O(n \log n)$ -bits, its efficiency is no better than a one-time pad. In our scheme, the private-key F is about $\text{poly}(n)$ -bits long, and it corresponds to a group of public-keys $(s, \rho_{i,k}^0)$, where s and i are chosen randomly. As mentioned above, s is open but $k = F(s)$ is hidden by the one-time key i , so the adversary cannot compute F . Our private-key can be reused $2^{O(n)}$ times.

If we change our scheme to one similar to that in [5], letting k be the private-key, ρ_k^0 the public-key, then using the method in Theorems 2.4 and 3.1 of [8], we can obtain following result:

Theorem 6. *If we fix the key pair at (ρ_k^0, k) , the key pair can only be used t times, $t = o(n)$.*

Proof: We calculate $\|\frac{1}{2^{n-1}} \sum_k (\rho_k^0 - \rho_k^1) \otimes (\rho_k^0)^{\otimes t}\|_{tr}$. For simplicity, we calculate $\|\frac{1}{2^{n-1}} \sum_k \rho_k^0 \otimes (\rho_k^0)^{\otimes t} - (\frac{I}{2^n})^{\otimes t+1}\|_{tr}$ and use the triangle inequality.

$$\rho_k^0 = \frac{1}{2 \cdot 2^n} \sum_i 2(|i\rangle\langle i| + |i\rangle\langle i \oplus k|) = \frac{1}{2^n} \sum_i \sum_x |i\rangle\langle i \oplus xk|, \quad (21)$$

where $x \in \{0, 1\}$. Thus we have

$$\begin{aligned} & \left\| \frac{1}{2^{n-1}} \sum_k \left((\rho_k^0)^{\otimes t} - \left(\frac{I}{2^n} \right)^{\otimes t} \right) \right\|_{tr} \\ &= \frac{1}{2^{n-1} \cdot 2^{nt}} \left\| \sum_k \sum_{i_1, \dots, i_t} \sum_{x_1, \dots, x_t} (|i_1, \dots, i_t\rangle\langle i_1 \oplus x_1 k, \dots, i_t \oplus x_t k| - \right. \\ & \quad \left. - |i_1, \dots, i_t\rangle\langle i_1, \dots, i_t|) \right\|_{tr} \\ &= \frac{1}{2^{n-1} \cdot 2^{nt}} \left\| \sum_k \sum_{i_1, \dots, i_t} \sum_{\substack{x_1, \dots, x_t \\ (x_1, \dots, x_t) \neq (0, \dots, 0)}} |i_1, \dots, i_t\rangle\langle i_1 \oplus x_1 k, \dots, i_t \oplus x_t k| \right\|_{tr} \\ &\leq \frac{1}{2^{n-1} \cdot 2^{nt}} \sum_{i_1, \dots, i_t} \| |i_1, \dots, i_t\rangle \| \cdot \left\| \sum_k \sum_{\substack{x_1, \dots, x_t \\ (x_1, \dots, x_t) \neq (0, \dots, 0)}} |i_1 \oplus x_1 k, \dots, i_t \oplus x_t k\rangle \right\| \\ &= \frac{1}{2^{n-1} \cdot 2^{nt}} \cdot 2^{nt} \cdot \sqrt{2^{n-1}(2^t - 1)} \\ &< \sqrt{\frac{1}{2^{n-t}}}. \end{aligned} \quad (22)$$

If we want $\|\frac{1}{2^{n-1}} \sum_k (\rho_k^0 - \rho_k^1) \otimes (\rho_k^0)^{\otimes t}\|_{tr} < 1/p(n)$, t should be $o(n)$. \square

Our QPKE is information theoretic security, which is realized via a new public-key algorithm structure. The security of the scheme in [8] is bounded information theoretic secure because it is based on a common public-key structure.

4. Extended QPKE for Multibits

We take a two-bit scheme as an example to show how to extend our QPKE to encrypt more than one bit with each pair of the public-key.

4.1. Four States Used to Construct the Public-Key

Definition 7. Define the n -qubit state as:

$$\begin{aligned} |\Psi_{k_1, k_2, i}^{00}\rangle &= \frac{1}{2}(|i\rangle + |i \oplus k_1\rangle + |i \oplus k_2\rangle + |i \oplus k_1 \oplus k_2\rangle) \\ &= \frac{1}{2}(|i_1\rangle|i_2\rangle + |i_1 \oplus k_{11}\rangle|i_2 \oplus k_{12}\rangle + |i_1 \oplus k_{21}\rangle|i_2 \oplus k_{22}\rangle \\ &\quad + |i_1 \oplus k_{11} \oplus k_{21}\rangle|i_2 \oplus k_{12} \oplus k_{22}\rangle), \end{aligned} \quad (23)$$

where $k_1, k_2, i \in Z_{2^n}$, $i_1, i_2 \in Z_{2^{\frac{n}{2}}}$, $k_{11}, k_{22} \in \Omega_{2^{\frac{n}{2}}}$, $k_{12}, k_{21} \in \Pi_{2^{\frac{n}{2}}}$, and $i = (i_1, i_2)$, $k_1 = (k_{11}, k_{12})$, $k_2 = (k_{21}, k_{22})$.

Applying $I^{\otimes \frac{n}{2}} \otimes Z^{\otimes \frac{n}{2}}$ on $|\Psi_{k_1, k_2, i}^{00}\rangle$, we obtain:

$$|\Psi_{k_1, k_2, i}^{01}\rangle = \frac{1}{2}(|i\rangle + |i \oplus k_1\rangle - |i \oplus k_2\rangle - |i \oplus k_1 \oplus k_2\rangle). \quad (24)$$

Applying $Z^{\otimes \frac{n}{2}} \otimes I^{\otimes \frac{n}{2}}$ on $|\Psi_{k_1, k_2, i}^{00}\rangle$, we obtain:

$$|\Psi_{k_1, k_2, i}^{10}\rangle = \frac{1}{2}(|i\rangle - |i \oplus k_1\rangle + |i \oplus k_2\rangle - |i \oplus k_1 \oplus k_2\rangle). \quad (25)$$

Applying $Z^{\otimes n}$ on $|\Psi_{k_1, k_2, i}^{00}\rangle$, we obtain:

$$|\Psi_{k_1, k_2, i}^{11}\rangle = \frac{1}{2}(|i\rangle - |i \oplus k_1\rangle - |i \oplus k_2\rangle + |i \oplus k_1 \oplus k_2\rangle). \quad (26)$$

Let the four states be the cipher text of two classical bits. We construct a two-bit oriented QPKE scheme based on them.

4.2. Trapdoor Property

Suppose k_1 and k_2 are given, for which without i the four mixed states are:

$$\rho_{k_1, k_2}^{00} = \frac{1}{2^n} \sum_i |\Psi_{k_1, k_2, i}^{00}\rangle \langle \Psi_{k_1, k_2, i}^{00}|, \quad (27)$$

$$\rho_{k_1, k_2}^{01} = \frac{1}{2^n} \sum_i |\Psi_{k_1, k_2, i}^{01}\rangle \langle \Psi_{k_1, k_2, i}^{01}|, \quad (28)$$

$$\rho_{k_1, k_2}^{10} = \frac{1}{2^n} \sum_i |\Psi_{k_1, k_2, i}^{10}\rangle \langle \Psi_{k_1, k_2, i}^{10}|, \quad (29)$$

$$\rho_{k_1, k_2}^{11} = \frac{1}{2^n} \sum_i |\Psi_{k_1, k_2, i}^{11}\rangle \langle \Psi_{k_1, k_2, i}^{11}|. \quad (30)$$

We take ρ_{k_1, k_2}^{10} as an example to explain that the algorithm described in Lemma 3 can also be used to distinguish these four states. The process includes the following steps:

- (1) Prepare two quantum registers, the first one contains two control bits in state $|0\rangle\langle 0| \otimes |0\rangle\langle 0|$, and the second one contains the unknown state. Take ρ_{k_1, k_2}^{10} as an example, then the state of the system is

$$|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes \rho_{k_1, k_2}^{10}. \quad (31)$$

- (2) Apply the Hadamard transformation to the first control bit and the controlled- k_1 operator to ρ_{k_1, k_2}^{10} , then the state of the system becomes

$$\frac{1}{2^n} \sum_i |\varphi_{k_1, k_2, i}^{10}\rangle \langle \varphi_{k_1, k_2, i}^{10}|, \quad (32)$$

where

$$\begin{aligned} |\varphi_{k_1, k_2, i}^{10}\rangle = \frac{1}{2\sqrt{2}} & \quad [|0\rangle|0\rangle(|i\rangle - |i \oplus k_1\rangle + |i \oplus k_2\rangle - |i \oplus k_1 \oplus k_2\rangle) \\ & + |1\rangle|0\rangle(|i \oplus k_1\rangle - |i\rangle - |i \oplus k_1 \oplus k_2\rangle + |i \oplus k_2\rangle)] \end{aligned} \quad (33)$$

- (3) Apply the Hadamard transformation to the first control bit again, and the state of the system becomes

$$|1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes \rho_{k_1, k_2}^{10}. \quad (34)$$

- (4) It can be seen that if we perform operations $H \otimes I \cdot C_{k_2} \cdot H \otimes I$ that is related to the second control bit, the final state of the system will be

$$|1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes \rho_{k_1, k_2}^{10}. \quad (35)$$

If the unknown state is one of the other three, the final state will be:

$$|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes \rho_{k_1, k_2}^{00}, \quad (36)$$

or

$$|0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes \rho_{k_1, k_2}^{01}, \quad (37)$$

or

$$|1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes \rho_{k_1, k_2}^{11}. \quad (38)$$

We can distinguish between these four states by measuring the first register.

4.3. Indistinguishability Property

Without k_1, k_2 and i , the ciphertext consists of four mixed states:

$$\rho_{odd}^{00} = \frac{1}{2^{2n-4}} \sum_{k_1, k_2} \rho_{k_1, k_2, i}^{00}, \quad (39)$$

$$\rho_{odd}^{01} = \frac{1}{2^{2n-4}} \sum_{k_1, k_2} \rho_{k_1, k_2, i}^{01}, \quad (40)$$

$$\rho_{odd}^{10} = \frac{1}{2^{2n-4}} \sum_{k_1, k_2} \rho_{k_1, k_2, i}^{10}, \quad (41)$$

$$\rho_{odd}^{11} = \frac{1}{2^{2n-4}} \sum_{k_1, k_2} \rho_{k_1, k_2, i}^{11}, \quad (42)$$

where k_1 and k_2 satisfy definition 7.

We can prove that the trace distance between any two of the four states is $O(\frac{1}{2^n})$.

- (1) The trace distance between ρ_{odd}^{00} and ρ_{odd}^{11} is $D(\rho_{odd}^{00}, \rho_{odd}^{11}) = \frac{1}{2^{n-2}}$ (see Appendix A).
- (2) The trace distance between ρ_{odd}^{00} and ρ_{odd}^{01} is $D(\rho_{odd}^{00}, \rho_{odd}^{01}) = \frac{1}{2^{n-2}}$ (see Appendix B).
- (3) The trace distance between ρ_{odd}^{00} and ρ_{odd}^{10} is $D(\rho_{odd}^{00}, \rho_{odd}^{10}) = \frac{1}{2^{n-2}}$. The proof is similar as that of $D(\rho_{odd}^{00}, \rho_{odd}^{01})$.
- (4) By the triangle inequality of the trace distance [11], we have

$$D(\rho_{odd}^{10}, \rho_{odd}^{01}) \leq D(\rho_{odd}^{00}, \rho_{odd}^{10}) + D(\rho_{odd}^{00}, \rho_{odd}^{01}) = \frac{1}{2^{n-3}}, \quad (43)$$

$$D(\rho_{odd}^{10}, \rho_{odd}^{11}) \leq D(\rho_{odd}^{00}, \rho_{odd}^{10}) + D(\rho_{odd}^{00}, \rho_{odd}^{11}) = \frac{1}{2^{n-3}}, \quad (44)$$

$$D(\rho_{odd}^{01}, \rho_{odd}^{11}) \leq D(\rho_{odd}^{00}, \rho_{odd}^{01}) + D(\rho_{odd}^{00}, \rho_{odd}^{11}) = \frac{1}{2^{n-3}}. \quad (45)$$

As shown in Figure 2, each trace distance between any two of these four states is $O(\frac{1}{2^n})$.

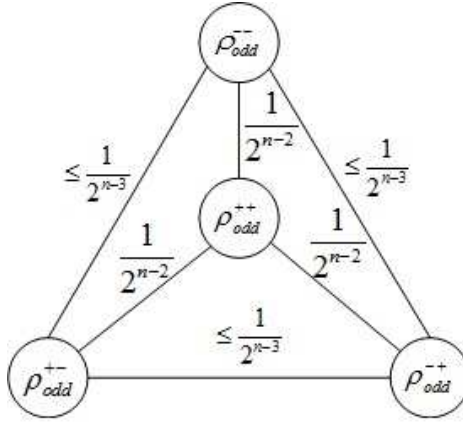


Figure 2: Trace distances between ρ_{odd}^{00} and other three states are $\frac{1}{2^{n-2}}$, so the trace distances between any two states of the other three are no more than $\frac{1}{2^{n-3}}$.

4.4. Extended QPKE for Two Bits

To extend the QPKE scheme to the two-bit oriented one, two aspects will be modified:

- (1) Bob chooses $s \in \{0, 1\}^{poly(n)}$ randomly, satisfies $F(s) = (k_1, k_2)$, where k_1, k_2 are defined above. Bob generates the n -qubit state $\rho_{k_1, k_2, i}^{00}$, and sends $(s, \rho_{k_1, k_2, i}^{00})$ to the public register as his public-key.
- (2) Alice encrypts 00 into $\rho_{k_1, k_2, i}^{00}$, 01 into $\rho_{k_1, k_2, i}^{01}$, 10 into $\rho_{k_1, k_2, i}^{10}$ and 11 into $\rho_{k_1, k_2, i}^{11}$, with operations $I^{\otimes n}$, $I^{\otimes \frac{n}{2}} \otimes Z^{\otimes \frac{n}{2}}$, $Z^{\otimes \frac{n}{2}} \otimes I^{\otimes \frac{n}{2}}$ and $Z^{\otimes n}$ respectively.

Because the trace distance of every pair of the four states is $O(\frac{1}{2^n})$, the QPKE scheme satisfies Eq.(2), so it is a scheme with information theoretic security under CPA.

4.5. Extended QPKE for Multi Bits

We now extend the QPKE to encrypt l bits. Define n -qubit state as:

$$|\Psi_{k_1, k_2, \dots, k_l, i}^{00 \dots 0}\rangle = \frac{1}{\sqrt{2^l}} \sum_{x_1, \dots, x_l} |i \oplus x_1 k_1 \oplus x_2 k_2 \dots \oplus x_l k_l\rangle. \quad (46)$$

where $k_1, k_2, \dots, k_l, i \in Z_{2^n}$, $x_1, \dots, x_l \in \{0, 1\}$, each k_j ($j = 1, \dots, l$) can be divided into l parts $k_j = (k_{j1}, \dots, k_{jl})$, only $W_H(k_{jj}) = \text{odd}$, and others are even.

We can use

$$\bigotimes_{j=1}^l ((1 - x_j)I + x_j Z)^{\otimes \frac{n}{l}} |\Psi_{k_1, k_2, \dots, k_l, i}^{00 \dots 0}\rangle, \quad (47)$$

to represent classical bits (x_1, \dots, x_l) . If we use these 2^l states to construct QPKE, the algorithm introduced in Lemma 3 can also be used for decryption. We conjecture that the trace distance of every pair of these mixed states is $O(\frac{1}{2^{n-l}})$, then the extended scheme is also one with information theoretic security under CPA.

5. Conclusions

We have proposed a definition for the information theoretic security of a quantum public-key encryption scheme, and proved the sufficient condition that a QPKC scheme has information theoretic security if the trace distance between every pair of ciphertext states is less than $1/p(n)$ for every positive polynomial $p(\cdot)$. We present bit-oriented and two-bit-oriented QPKE schemes with a new algorithm structure, and prove that both of them satisfy our security definition. Finally, we extend the QPKE to the multi-bit-oriented case, and conjecture that the scheme is also one with information

theoretic security. The information theoretic security of our QPKE schemes depends directly on the new structure of the public-key algorithm that we have introduced here.

Acknowledgements

This work was supported by the National Natural Science Foundation of China under Grant No. 60573051.

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Appendix A.

Trace Distance between ρ_{odd}^{00} and ρ_{odd}^{11} :

$$\begin{aligned} \rho_{odd}^{00} - \rho_{odd}^{11} &= \frac{2}{2^{2n-4} \cdot 2^n \cdot 4} \sum_{k_1, k_2, i} (E_{i, i \oplus k_1} + E_{i, i \oplus k_2} + E_{i \oplus k_1, i} + E_{i \oplus k_1, i \oplus k_1 \oplus k_2} \\ &\quad + E_{i \oplus k_2, i} + E_{i \oplus k_2, i \oplus k_1 \oplus k_2} + E_{i \oplus k_1 \oplus k_2, i \oplus k_1} + E_{i \oplus k_1 \oplus k_2, i \oplus k_2}), \end{aligned} \quad (A.1)$$

since

$$\sum_i E_{i, i \oplus k_1} = \sum_i E_{i \oplus k_1, i} = \sum_i E_{i \oplus k_2, i \oplus k_1 \oplus k_2} = \sum_i E_{i \oplus k_1 \oplus k_2, i \oplus k_2}, \quad (A.2)$$

and

$$\sum_i E_{i, i \oplus k_2} = \sum_i E_{i \oplus k_1, i \oplus k_1 \oplus k_2} = \sum_i E_{i \oplus k_2, i} = \sum_i E_{i \oplus k_1 \oplus k_2, i \oplus k_1}, \quad (A.3)$$

then

$$\begin{aligned} \rho_{odd}^{00} - \rho_{odd}^{11} &= \frac{8}{2^{2n-4} \cdot 2^n \cdot 4} \sum_{k_1, k_2, i} (E_{i, i \oplus k_1} + E_{i, i \oplus k_2}) \\ &= \frac{8 \cdot 2^{n-2}}{2^{2n-4} \cdot 2^n \cdot 4} \sum_i \left(\sum_{k_1} E_{i, i \oplus k_1} + \sum_{k_2} E_{i, i \oplus k_2} \right) \\ &= \frac{1}{2^{n-3} \cdot 2^n} \sum_i \sum_{j \in \Omega_n} E_{i, i \oplus j} \\ &= \frac{1}{2^{n-3} \cdot 2^n} A_{odd}. \end{aligned} \quad (A.4)$$

where the A_{odd} is the same as in (14), according to (16) we have

$$D(\rho_{odd}^{00}, \rho_{odd}^{11}) = \frac{1}{2} tr |\rho_{odd}^{00} - \rho_{odd}^{11}| = \frac{1}{2^{n-2} \cdot 2^n} tr |A_{odd}| = \frac{1}{2^{n-2}}. \quad (A.5)$$

Appendix B.

Trace Distance between ρ_{odd}^{00} and ρ_{odd}^{01} :

$$\begin{aligned} \rho_{odd}^{00} - \rho_{odd}^{01} &= \frac{2}{2^{2n-4} \cdot 2^n \cdot 4} \sum_{k_1, k_2, i} (E_{i, i \oplus k_2} + E_{i, i \oplus k_1 \oplus k_2} + E_{i \oplus k_1, i \oplus k_2} + \\ &+ E_{i \oplus k_1, i \oplus k_1 \oplus k_2} + E_{i \oplus k_2, i} + E_{i \oplus k_2, i \oplus k_1} + E_{i \oplus k_1 \oplus k_2, i} + E_{i \oplus k_1 \oplus k_2, i \oplus k_1}), \end{aligned} \quad (B.1)$$

since

$$\sum_i E_{i, i \oplus k_2} = \sum_i E_{i \oplus k_1, i \oplus k_1 \oplus k_2} = \sum_i E_{i \oplus k_2, i} = \sum_i E_{i \oplus k_1 \oplus k_2, i \oplus k_1}, \quad (B.2)$$

and

$$\sum_i E_{i, i \oplus k_1 \oplus k_2} = \sum_i E_{i \oplus k_1, i \oplus k_2} = \sum_i E_{i \oplus k_2, i \oplus k_1} = \sum_i E_{i \oplus k_1 \oplus k_2, i}, \quad (B.3)$$

then

$$\begin{aligned} &\rho_{odd}^{00} - \rho_{odd}^{01} \\ &= \frac{8}{2^{2n-4} \cdot 2^n \cdot 4} \sum_{k_1, k_2, i} (E_{i, i \oplus k_2} + E_{i, i \oplus k_1 \oplus k_2}) \\ &= \frac{8}{2^{3n-2}} \sum_{\substack{k_{11}, k_{12}, k_{21}, \\ k_{22}, i_1, i_2}} (E_{i_1, i_1 \oplus k_{21}} \otimes E_{i_2, i_2 \oplus k_{22}} + E_{i_1, i_1 \oplus k_{11} \oplus k_{21}} \otimes E_{i_2, i_2 \oplus k_{12} \oplus k_{22}}). \end{aligned} \quad (B.4)$$

Since $W_H(k_{11}) = odd$, $W_H(k_{12}) = even$, $W_H(k_{21}) = even$ and $W_H(k_{22}) = odd$, we have

$$\sum_{k_{11}} E_{i_1, i_1 \oplus k_{11} \oplus k_{21}} = \sum_{k_{11}} E_{i_1, i_1 \oplus k_{11}}, \quad (B.5)$$

$$\sum_{k_{22}} E_{i_2, i_2 \oplus k_{12} \oplus k_{22}} = \sum_{k_{22}} E_{i_2, i_2 \oplus k_{22}}. \quad (B.6)$$

then

$$\begin{aligned}
\rho_{odd}^{00} - \rho_{odd}^{01} &= \frac{8 \cdot 2^{n-2}}{2^{3n-2}} \sum_{i_1, i_2} \left(\sum_{k_{21}, k_{22}} E_{i_1, i_1 \oplus k_{21}} \otimes E_{i_2, i_2 \oplus k_{22}} + \sum_{k_{11}, k_{22}} E_{i_1, i_1 \oplus k_{11}} \otimes E_{i_2, i_2 \oplus k_{22}} \right) \\
&= \frac{1}{2^{2n-3}} \left(\sum_{i_1, k_{21}} E_{i_1, i_1 \oplus k_{21}} + \sum_{i_1, k_{11}} E_{i_1, i_1 \oplus k_{11}} \right) \otimes \left(\sum_{i_2, k_{22}} E_{i_2, i_2 \oplus k_{22}} \right) \\
&= \frac{1}{2^{2n-3}} A_{\frac{n}{2}} \otimes A_{odd, \frac{n}{2}}, \tag{B.7}
\end{aligned}$$

where $A_{odd, \frac{n}{2}}$ is a $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$ matrix similar to A_{odd} , and

$$A_{\frac{n}{2}} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]^{\otimes \frac{n}{2}},$$

then we have

$$\begin{aligned}
tr|A_{\frac{n}{2}}| &= 2^{\frac{n}{2}}, \\
tr|A_{odd, \frac{n}{2}}| &= 2^{\frac{n}{2}},
\end{aligned}$$

$$D(\rho_{odd}^{00}, \rho_{odd}^{01}) = \frac{1}{2} tr|\rho_{odd}^{00} - \rho_{odd}^{01}| = \frac{1}{2^{2n-2}} tr|A_{\frac{n}{2}}| \times tr|A_{odd, \frac{n}{2}}| = \frac{1}{2^{n-2}}.$$